

Anomalous Dimension of the Electrical Current in the Normal State of the Cuprates from the Fractional Aharonov-Bohm Effect

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We show here that if the current in the normal state of the cuprates has an anomalous dimension, then the Aharonov-Bohm flux through a ring does not have the standard eBA/\hbar form, where A is the area and B the external magnetic field, but instead is modified by a geometrical factor that depends directly on the anomalous dimension of the current. We calculate the signal in square and disk geometries. In both cases, the deviation from the standard result is striking and offers a fingerprint about what precisely is strange about the strange metal.

High-temperature superconductivity in the cuprates remains an unsolved problem because no knock-down experiment has revealed unambiguously the nature of the charge carriers in the normal state. What we know for sure is that the standard theory of metals and a single-parameter[1, 2] formulation of quantum criticality cannot simultaneously explain T -linear resistivity, power-law optical conductivity, $\omega^{-2/3}$ [3, 4], breakdown of the Weidemann-Franz law[5], and the scaling of the Hall angle[6]. Recent theoretical work[2, 7, 8] suggests that the strangeness of the strange metal regime is that the current possesses an anomalous dimension.

Indeed, this is a striking prediction because a textbook problem[9–11] in quantum field theory is to prove that conserved quantities cannot acquire anomalous dimensions under renormalization. In a relativistic theory, this rule fixes the dimension of the current to be $d - 1$ as the vector potential is dimension one. For a local theory away from the strict relativistic limit, the dimension of the current can change by two mechanisms: 1) reduction of the effective dimensionality, that is a violation of hyperscaling[12, 13] with exponent θ or 2) space and time scale differently thereby requiring a dynamical exponent $z > 1$. While the new scaling of the current is now $d - \theta + z - 1$, neither of these scenarios violates the Ward identities[9, 10] that underlie the proof prohibiting anomalous dimensions from occurring in a conserved current. A third mechanism for the emergence of an anomalous dimension for the current is that the underlying theory is inherently non-local. It is this mechanism that appears to be operative in the recent work[7, 8] which showed that extending the single-parameter quantum critical scenario[2] to include a multi-band or unparticle sector with a running charge[14] leads to a consistent explanation of all the power laws experimentally

observed in the dc[5, 6, 15, 16] and ac[3, 4] transport properties in the strange metal phase of the cuprates. A running charge is possible only if the vector potential acquires an anomalous dimension, Φ . To fit the cuprates $\Phi = 4/3$ [2, 8]. An unparticle sector is inherently non-local and such non-locality offers a loop-hole around the Ward identities that prohibit the current from acquiring an anomalous dimension. Given the novelty of an electric current acquiring an anomalous dimension as the unique underlying feature of the strange metal, it would be ideal to design an experiment, not tethered to any scaling analysis, that can critically test this idea unambiguously. Should this be borne out experimentally, then the normal state of the cuprates would represent the first example in nature of current carrying excitations with an anomalous dimension.

In this paper, we propose such an experiment. Since it is the vector potential that communicates the anomalous dimension to the electrical current, this effect should be detectable from a simple Aharonov-Bohm[17] (AB) experiment in the strange metal regime. We show here that the AB phase picks up a factor that depends on the anomalous dimension and hence provides an unambiguous fingerprint of the non-locality of the current. This effect obtains because when the vector potential has an anomalous dimension, the standard AB phase, $\Delta\phi = eBA/\hbar$, is no longer dimensionless. As a result, what replaces the AB phase has to include an extra factor of L^{α_B-2} , where L has units of length and α_B is the scaling dimension of the B-field, for $\Delta\phi$ to be dimensionless. We calculate this effect explicitly.

Since our emphasis is the electrical response, we need focus solely on the part of the action

$$S \rightarrow S_{\text{stuff}} + \int d^d x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu \right) \quad (1)$$

that couples to the electromagnetic gauge field. Here $F_{\mu\nu}$ is the field strength, A_μ is the vector potential, and J^μ is the current. The electromagnetic gauge

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enters the Schrödinger equation through the covariant derivative, $D_\mu = \partial_\mu - ieA_\mu/\hbar$ thereby demanding that $[A_\mu] = 1$. Going beyond this constraint requires a redefinition of the gauge transformation [18–20]

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu^{\alpha_\mu} \Lambda(x) \quad (2)$$

which contains fractional derivative, $\partial_\mu^{\alpha_\mu}$, of order α_μ . The corresponding field strength is redefined [18–20] as

$$F_{\mu\nu}^\alpha = \partial_\mu^{\alpha_\mu} A_\nu - \partial_\nu^{\alpha_\nu} A_\mu. \quad (3)$$

We set $\alpha_i = \alpha$ for simplicity and, to maintain causality[18], set $\alpha_t = 1$. Despite these changes, we still implement the usual $U(1)$ transformation on the matter field, $\psi(x) \rightarrow \exp(i\frac{e}{\hbar}\Lambda(x))\psi(x)$. From the fractional gauge transformation, it follows that $[A_i] = \alpha$ and $\Phi = 1 - \alpha$.

Computing the equations of motion requires some convention on the definition of fractional calculus. Such machinery will facilitate extending the standard integer derivatives and integrals to those involving fractional powers

$$\{I_x^n, \frac{\partial^n}{\partial x^n}\} \rightarrow \{I_x^\alpha, \frac{\partial^\alpha}{\partial x^\alpha}\}. \quad (4)$$

Here I_x^n is defined as a repeated integral n times over x . We focus on four definitions of fractional calculi: Left and Right Liouville, Feller, and Riesz[19, 21–24]. These definitions of fractional calculi can be formulated in real or Fourier space. For our purposes, it is most useful to implement the Fourier-space formulation,

$$\partial_x^\alpha f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F(\alpha, k) \tilde{f}(k) \quad (5)$$

$$I_x^\alpha f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} F^{-1}(\alpha, k) \tilde{f}(k), \quad (6)$$

where

$$F(\alpha, k) = \begin{cases} (ik)^\alpha & \text{Right Liouville} \\ (-ik)^\alpha & \text{Left Liouville} \\ i\text{sgn}(k)|k|^\alpha & \text{Feller} \\ |k|^\alpha & \text{Riesz}, \end{cases} \quad (7)$$

and ∂_x^α and I_x^α denote the fractional derivative and integral. The convention of the branch cut we use is $-\pi < \theta \leq \pi$. Left and right Liouville are spatially asymmetric because, in real space, the operations involve an integration on the left and on the right of x , respectively (Eqs. (A1) - (A4)). Feller calculus is odd under parity, and thus it resembles

an odd-integer-order calculus. On the other hand, since Riesz calculus is even under parity, its behavior is similar to an even-integer-order calculus. We will use all four of these definitions to derive the AB phase with a gauge field that possesses an anomalous dimension.

We can now formulate the equations of motion. To facilitate such a derivation, we need the fractional version,

$$\int \psi_1^* \partial^\alpha \psi_2 = \int \tilde{\partial}^\alpha \psi_1^* \psi_2, \quad (8)$$

of integration by parts, where $\tilde{\partial}^\alpha$ is the conjugate of ∂^α . If ∂^α is the left (right) Liouville derivative, its conjugate is the right (left) Liouville derivative [24, 25]. Furthermore, $\tilde{\partial}_F^\alpha = -\partial_F^\alpha$ and $\tilde{\partial}_{RZ}^\alpha = \partial_{RZ}^\alpha$ ¹. Here the subscripts F and RZ correspond to Feller and Riesz. Using this integration by parts property, one can derive the equations of motion from this action [18] as

$$\tilde{\partial}_\mu^{\alpha_\mu} F^{\mu\nu} = J^\nu. \quad (9)$$

The Bianchi identity,

$$\partial_\mu^{\alpha_\mu} \tilde{F}^{\mu\nu} = 0, \quad (10)$$

arises from the antisymmetry in F (Eq. (3)) where $\tilde{F}^{\mu\nu} \equiv \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$. We identify the fractional electric and magnetic fields as $E^i = -F^{0i}$ and $\epsilon^{ijk} B^k = -F^{ij}$ respectively. In 3+1 dimensions, all the Maxwell equations are modified accordingly with fractional derivatives, for example

$$\nabla^\alpha \cdot \mathbf{B} = 0 \quad (11)$$

where $\nabla^\alpha \equiv \{\partial_x^\alpha, \partial_y^\alpha, \partial_z^\alpha\}$. Because $\nabla^\alpha \times \mathbf{A} = \mathbf{B}$ and $[A_i] = \alpha$, the units of the magnetic field are $[B] = 2\alpha$. We will work with the convention that it is the B -field that has the anomalous dimension and the charge remains unchanged. A further consequence of the Maxwell equations is that the continuity equation[18]

$$\nabla^\alpha \cdot \mathbf{J} + \partial_t \rho = 0 \quad (12)$$

must involve the fractional derivative. It is this version of the continuity equation that must be used instead of the traditional one[2] in the context of a current with an anomalous dimension.

We are now set up to calculate the AB phase using fractional calculus. Let us define the covariant

¹ The integration by parts for Feller and Riesz derivatives follows from the integration by parts for Liouville derivatives and Eqs. A9 and A11.

derivative $D_i \equiv \partial_i - i\frac{e}{\hbar}a_i$ with the associated gauge connection [19]

$$a_i \equiv [\partial_i, I_i^\alpha A_i] = \partial_i I_i^\alpha A_i \quad (13)$$

which transforms as $a_\mu \rightarrow a_\mu + \partial_\mu \Lambda$ because the fundamental theorem of fractional calculus[21] in any of the forms presented earlier guarantees that $I^\alpha \partial^\alpha \Lambda = \Lambda$. As a result, the Schrödinger Lagrangian,

$$\mathcal{L} = \frac{\hbar^2}{2m} D_i \psi^\dagger D_i \psi - i\hbar \psi^\dagger D_t \psi \quad (14)$$

is invariant under the $U(1)$ transformation. Choosing $A_0 = 0$, we reduce the Schrödinger equation to

$$-\frac{\hbar^2}{2m} (\partial_i - i\frac{e}{\hbar}a_i)^2 \psi = i\hbar \partial_t \psi. \quad (15)$$

To derive the AB phase, let us consider a particle confined on the x, y plane with fractional magnetic field applied along the z axis. Assume a particle can move from point r_i to r_f along path 1 (with wave function ψ_1) and along path 2 (with wave function ψ_2). The total wave function at the point r_f at zero fractional magnetic field ($a_i = 0$) is $\psi = \psi_1 + \psi_2$. The total wave function at r_f changes to

$$\begin{aligned} \psi &= \exp(i\frac{e}{\hbar} \int_{\text{path 1}} \mathbf{a}(r') \cdot d\mathbf{l}') \psi_1(r_f, t) \\ &+ \exp(i\frac{e}{\hbar} \int_{\text{path 2}} \mathbf{a}(r') \cdot d\mathbf{l}') \psi_2(r_f, t) \\ &= C \left(\psi_1(r_f, t) + \exp(i\frac{e}{\hbar} \oint \mathbf{a}(r') \cdot d\mathbf{l}') \psi_2(r_f, t) \right). \end{aligned} \quad (16)$$

Here C is an over all phase factor $= \exp(i\frac{e}{\hbar} \int_{\text{path 1}} \mathbf{a}(r') \cdot d\mathbf{l}')$. The phase difference between the two paths due to the gauge field is

$$\Delta\phi = \frac{e}{\hbar} \oint \mathbf{a}(r') \cdot d\mathbf{l}' \quad (18)$$

To compute this Aharonov-Bohm phase, we consider two different geometries. In Fig. (1), motion is constrained to move along the rectangularly shaped region shown. The fractional magnetic field is applied along z -axis and is confined to the region of size $\ell \times \ell$ indicated inside the rectangle. In real space, \mathbf{B} takes on the form

$$\mathbf{B}(x, y) = B\Theta(\ell^2/4 - x^2)\Theta(\ell^2/4 - y^2)\hat{z} \quad (19)$$

which in momentum space is given by

$$\mathbf{B}(\mathbf{k}) = B_z(\mathbf{k})\hat{z} = 4B \frac{\sin \frac{k_x \ell}{2} \sin \frac{k_y \ell}{2}}{k_x k_y} \hat{z}. \quad (20)$$

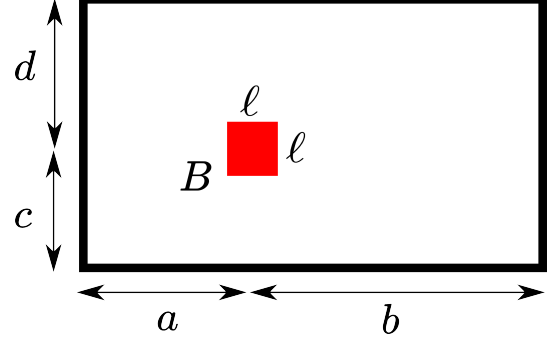


FIG. 1: Rectangle geometry that confines particle motion. The fractional magnetic field is confined to the red region of size ℓ in the figure.

To obtain the AB phase we first solve $\mathbf{B}(k) = \{F(\alpha, k_x), F(\alpha, k_y), 0\} \times \mathbf{A}(\mathbf{k})$ for $\mathbf{A}(\mathbf{k})$:

$$\mathbf{A}(\mathbf{k}) = \frac{B_z(\mathbf{k})\{-F(\alpha, k_y), F(\alpha, k_x), 0\}}{F^2(\alpha, k_x) + F^2(\alpha, k_y)}. \quad (21)$$

Here the function $F(\alpha, k_i)$ is defined in Eq. (7). All of the different ways of defining the fractional integral come into play in the evaluation of Eq. (13). In evaluating the subsequent expressions, we find it easiest to work with $\mathbf{b} = i\mathbf{k} \times \mathbf{a}$ which can be written,

$$b_{z,i} = 4B\ell^{2\alpha-2} f_i\left(\frac{x}{\ell}\right) f_i\left(\frac{y}{\ell}\right), \quad (22)$$

in terms of the function

$$f_i(s) = \int_{-\infty}^{\infty} \frac{dz}{2\pi} g_i(z) e^{izs} \sin \frac{z}{2} \quad (23)$$

where $g_{1(2)}(z) = i(\pm iz)^{-\alpha}$ for the left (+) and right (-) Liouville calculi, $g_3(z) = \text{sgn}(z)|z|^{-\alpha}$ for the Feller calculus and $g_4(z) = |z|^{-\alpha}$ for the Riesz calculus. The integral of $b_z(x, y)$,

$$\Delta\phi_R = \frac{e}{\hbar} \int_{-a}^b dx dy \int_{-c}^d b_{z,i}(x, y) \quad (24)$$

in the x, y plane determines the AB phase. Evaluating all of the integrals, we see that regardless of the fractional calculus used, the AB phase

$$\Delta\phi_R = \begin{cases} \frac{eB\ell^2}{\hbar} \left(\frac{b^{\alpha-1}d^{\alpha-1}}{\Gamma^2(\alpha)} \right) & b \gg \ell, d \gg \ell \\ \frac{eB\ell^2}{\hbar} \left(\frac{a^{\alpha-1}c^{\alpha-1}}{\Gamma^2(\alpha)} \right) & a \gg \ell, c \gg \ell \\ \frac{eB\ell^2}{\hbar} \left(\frac{(a^{\alpha-1}+b^{\alpha-1})(c^{\alpha-1}+d^{\alpha-1})}{4\Gamma^2(\alpha) \sin^2 \frac{\pi\alpha}{2}} \right) & a, b, c, d \gg \ell \\ \frac{eB\ell^2}{\hbar} \left(\frac{(a^{\alpha-1}-b^{\alpha-1})(c^{\alpha-1}-d^{\alpha-1})}{4\Gamma^2(\alpha) \cos^2 \frac{\pi\alpha}{2}} \right) & a, b, c, d \gg \ell \end{cases} \quad (25)$$

picks up a geometric factor that is directly determined by the anomalous dimension α of the vector potential. The four results are listed in order of the fractional calculus defined in Eq. (7). The limiting values, except the one from Riesz calculus, are $eB\ell^2/\hbar$ as $\alpha \rightarrow 1$. Riesz calculus has even parity, so one cannot expect it to have the same behavior as the first order derivative. Since both Liouville calculi are spatially asymmetric, their AB phases are asymmetric. The convention that we have used is that the anomalous dimension is carried by the B-field not the charge such that $[B] = 2\alpha$. As a result $\Delta\phi$ is dimensionless.

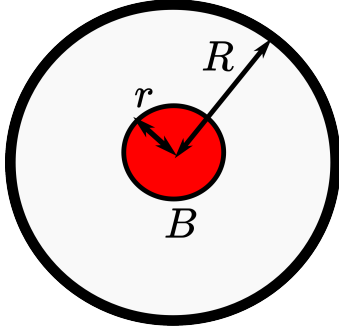


FIG. 2: Disk geometry for AB phase calculation. The fractional magnetic field pierces the disk in a small region of radius, r .

Additionally, we consider the disk geometry which is probably more experimentally tractable, though more computationally challenging as the results can only be expressed in terms of special functions. The magnetic field is given by

$$\mathbf{B} = B_z(\boldsymbol{\rho})\hat{z} = \Theta(r - \rho)B\hat{z} \quad (26)$$

which can be Fourier transformed to

$$B_z(\mathbf{k}) = B \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-i\mathbf{k}\cdot\boldsymbol{\rho}} \Theta(r - \rho). \quad (27)$$

We then transform to polar coordinates, $\boldsymbol{\rho} = \rho \cos \phi \hat{x} + \rho \sin \phi \hat{y}$ and $\mathbf{k} = k \cos \xi \hat{x} + k \sin \xi \hat{y}$. The result is

$$\begin{aligned} B_z(\mathbf{k}) &= B \int_0^{2\pi} d\phi \int_0^r d\rho \rho e^{-ik\rho \cos(\phi - \xi)} \\ &= \frac{2\pi r}{k} B J_1(kr), \end{aligned} \quad (28)$$

where J_1 is a Bessel function. In analogy with the rectangular geometry, the Liouville magnetic kernel is

$$\begin{aligned} b_z(\mathbf{k}) &= B_z(\mathbf{k})(\pm i k_x)^{1-\alpha}(\pm i k_y)^{1-\alpha} \\ &= 2\pi r B k^{2-2\alpha} J_1(kr)(\pm i \cos \xi)^{1-\alpha}(\pm i \sin \xi)^{1-\alpha}, \end{aligned} \quad (29)$$

where the \pm refer to the left (+) and right (-) Liouville formulations. The phase difference is given by

$$\Delta\phi_D = \frac{e}{\hbar} \frac{1}{4\pi^2} \int_0^\infty dk \int_0^R d\rho \int_0^{2\pi} d\theta \rho e^{ik\rho \cos(\theta - \xi)} b_z(\mathbf{k}) \quad (30)$$

The result of all of these integrals are shown in Appendix (C 1) leading to

$$\begin{aligned} \Delta\phi_D &= \frac{e}{\hbar} \pi r^2 B R^{2\alpha-2} \left(\frac{\sqrt{\pi} 2^{1-\alpha} \Gamma(2-\alpha) \Gamma(1-\frac{\alpha}{2})}{\Gamma(\alpha) \Gamma(\frac{3}{2}-\frac{\alpha}{2})} \right. \\ &\quad \left. \times \sin^2 \frac{\pi\alpha}{2} {}_2F_1(1-\alpha, 2-\alpha; 2; \frac{r^2}{R^2}) \right) \end{aligned} \quad (31)$$

as the phase shift in the left and right Liouville formulations. Here ${}_2F_1(a, b; c; z)$ is a hypergeometric function and the terms in the parenthesis reduce to unity in the limit $\alpha \rightarrow 1$. In the Feller formulation, the magnetic kernel

$$b_z(\mathbf{k}) = 2\pi r B k^{1-2\alpha} J_1(kr) |\cos \xi|^{1-\alpha} |\sin \xi|^{1-\alpha}. \quad (32)$$

involves the magnitude of the trigonometric functions and consequently the integral

$$\int_0^{2\pi} d\xi |\cos \xi|^{1-\alpha} |\sin \xi|^{1-\alpha} = \frac{2^\alpha \sqrt{\pi} \Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{3}{2}-\frac{\alpha}{2})}. \quad (33)$$

does not contain the $\sin^2 \pi\alpha/2$ term. Consequently, the phase difference is identical to that in Liouville fractional calculi except the $\sin^2 \pi\alpha/2$ term is absent. For the Riesz formulation,

$$b_z(\mathbf{k}) = -2\pi r B k^{1-2\alpha} J_1(kr) \cos \xi |\cos \xi|^{-\alpha} \sin \xi |\sin \xi|^{-\alpha} \quad (34)$$

and as a consequence, the integral over ξ vanishes because $\cos \xi |\cos \xi|^{-\alpha} \sin \xi |\sin \xi|^{-\alpha}$ is an odd function. Hence, in this case, $\Delta\phi_D = 0$. This result is not surprising, because from Eq. (25), the AB phase from Riesz calculus when $a = b$ and $c = d$ is zero.

We have shown here that the presence of an anomalous dimension leads to a significant deviation from the standard AB phase. Appearing in the AB phase is a geometric factor in which the size of the sample is raised to a power involving the anomalous dimension. This extra sample-size dependence reflects the non-locality of the current. The correction is sizeable as it involves a ratio of the sample size to the region where the flux is threaded. As a result, we have provided an experimental diagnostic

that is independent of any scaling ansatz. Experiments performed on either the disk or rectangular geometries patterned out of cuprates in the strange metal regime should provide a clear falsification of the claim that what is strange about the strange metal is that the current possess an anomalous dimension.

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Appendix A: Fractional Calculus in Coordinate-Space Representation

Gauge fields with anomalous dimensions require fractional calculus. In terms of formal mathematical operations, the methods outlined have restrictions regarding the range of validity of α . For example either $0 < \alpha < 1$ or $0 < \alpha < 2$ for the Feller or Reisz calculi. Nonetheless, the results can be analytically continued outside this range.

The fractional calculi in the text (Eqs. (5) - (7)) are formulated in the Fourier-space representations. Alternatively, they can be defined in coordinate space. Let a and b be real numbers. We define the following notations for the fractional derivative and the corresponding integral for $x > a$:

$$D_a^x(\alpha)f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x dx' (x-x')^{n-\alpha} f(x') \quad (\text{A1})$$

$$I_a^x(\alpha)f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x dx' (x-x')^{\alpha-1} f(x'). \quad (\text{A2})$$

When $x < b$, we define

$$D_x^b(\alpha)f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b dx' (x'-x)^{n-\alpha} f(x') \quad (\text{A3})$$

$$I_x^b(\alpha)f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b dx' (x'-x)^{\alpha-1} f(x') \quad (\text{A4})$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α .

The Left Riemann-Liouville fractional calculus corresponds to

$$D_{LRL}^\alpha = D_a^x(\alpha) \quad (\text{A5})$$

$$I_{LRL}^\alpha = I_a^x(\alpha), \quad (\text{A6})$$

while the Right Riemann-Liouville fractional calculus is

$$D_{RRL}^\alpha = D_x^b(\alpha) \quad (\text{A7})$$

$$I_{RRL}^\alpha = I_x^b(\alpha). \quad (\text{A8})$$

The Liouville fractional calculi used in the text is the special case of the Riemann-Liouville calculi with $a = -\infty$ and $b = \infty$.

The Feller fractional calculus corresponds to

$$D_F^\alpha = \frac{1}{2 \sin \frac{\pi\alpha}{2}} (D_{-\infty}^x(\alpha) - D_x^\infty(\alpha)) \quad (\text{A9})$$

$$I_F^\alpha = \frac{1}{2 \sin \frac{\pi\alpha}{2}} (I_{-\infty}^x(\alpha) - I_x^\infty(\alpha)) \quad (\text{A10})$$

and the Riesz fractional calculus corresponds to

$$D_{RZ}^\alpha = \frac{1}{2 \cos \frac{\pi\alpha}{2}} (D_{-\infty}^x(\alpha) + D_x^\infty(\alpha)) \quad (\text{A11})$$

$$I_{RZ}^\alpha = \frac{1}{2 \cos \frac{\pi\alpha}{2}} (I_{-\infty}^x(\alpha) + I_x^\infty(\alpha)). \quad (\text{A12})$$

The Fourier-space formulations (Eqs. (5) - (7)) can be shown to be the same as the coordinate space representations (Eqs. (A5) - (A8) with $a = -\infty$ and $b = \infty$ and Eqs. (A9) - (A12)). We explicitly show this for the case of the left Liouville calculus. We start by rewriting Eqs. (6) and (7) in the case of left Liouville to

$$I_{LL}^\alpha f(x) = \int_{-\infty}^{\infty} dx' K(x-x') f(x') \quad (\text{A13})$$

where the kernel $K(x-x') = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x-x')} (ik)^{-\alpha}$ and the subscript LL denotes left Liouville. This integral can be evaluated to be

$$K(x-x') = \Theta(x-x') \frac{(x-x')^{\alpha-1}}{\Gamma(\alpha)} \quad (\text{A14})$$

when $0 < \alpha < 1$. Thus, the left Liouville integral in coordinate space is

$$I_{LL}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x dx' (x-x')^{\alpha-1} f(x') = I_{-\infty}^x(\alpha). \quad (\text{A15})$$

Similarly, we rewrite the left Liouville derivative from Eqs. (5) and (7) to

$$\begin{aligned} \partial_{LL}^\alpha f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} (ik)^\alpha \tilde{f}(k) \\ &= \frac{d^n}{dx^n} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} (ik)^{\alpha-n} \tilde{f}(k) \\ &= \frac{d^n}{dx^n} I_{LL}^{n-\alpha} f(x) \\ &= \frac{1}{\Gamma(\alpha)} \frac{d^n}{dx^n} \int_{-\infty}^x dx' (x-x')^{\alpha-n} f(x') \\ &= D_{-\infty}^x(\alpha) f(x) \end{aligned} \quad (\text{A16})$$

where $n = [\alpha] + 1$. The equivalences between the Fourier-space and the coordinate-space formulations of the right Liouville, Feller, and Riesz can be shown in similar manner.

Appendix B: Fractional Aharonov-Bohm Effect in Rectangular geometry

The expression for \mathbf{B} from Fig. (1) is

$$\mathbf{B}(x, y) = B \Theta(\ell^2/4 - x^2) \Theta(\ell^2/4 - y^2) \hat{z}. \quad (\text{B1})$$

The Fourier transform of $B(x, y)$ is

$$\mathbf{B}(\mathbf{k}) = B_z(\mathbf{k}) \hat{z} = 4B \frac{\sin \frac{k_x \ell}{2} \sin \frac{k_y \ell}{2}}{k_x k_y} \hat{z}. \quad (\text{B2})$$

Below we directly use the Fourier-space formulations to evaluate fractional derivatives and integrals.

1. Left Liouville Fractional Calculus

We solve $\mathbf{A}(\mathbf{k})$ from

$$\mathbf{B}(\mathbf{k}) = (i\mathbf{k})^\alpha \times \mathbf{A}(\mathbf{k}) \quad (\text{B3})$$

where $(i\mathbf{k})^\alpha = \{(ik_x)^\alpha, (ik_y)^\alpha, 0\}$. A choice of $\mathbf{A}(\mathbf{k})$ that satisfies Eq. (B3) is

$$\mathbf{A}(\mathbf{k}) = \frac{B_z(\mathbf{k})}{(ik_x)^{2\alpha} + (ik_y)^{2\alpha}} \{-(ik_y)^\alpha, (ik_x)^\alpha, 0\}. \quad (\text{B4})$$

Next, using Eq. (13), we obtain $\mathbf{a}(\mathbf{k})$ as

$$\mathbf{a}(\mathbf{k}) = \frac{B_z(\mathbf{k})}{(ik_x)^{2\alpha} + (ik_y)^{2\alpha}} \{-(ik_x)^{1-\alpha}(ik_y)^\alpha, (ik_x)^\alpha(ik_y)^{1-\alpha}, 0\}. \quad (\text{B5})$$

It is easiest to work with $\mathbf{b}(\mathbf{k}) = (i\mathbf{k}) \times \mathbf{a}(\mathbf{k})$. We obtain

$$\begin{aligned} \mathbf{b}(\mathbf{k}) &= B_z(\mathbf{k})(ik_x)^{1-\alpha}(ik_y)^{1-\alpha}\hat{z} \\ &= -4B \sin \frac{k_x \ell}{2} \sin \frac{k_y \ell}{2} (ik_x)^{-\alpha} (ik_y)^{-\alpha} \hat{z}, \end{aligned} \quad (\text{B6})$$

which in position space becomes

$$\begin{aligned} b_z(x, y) &= \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} b_z(\mathbf{k}) \\ &= 4B\ell^{2\alpha-2} f_1\left(\frac{x}{\ell}\right) f_1\left(\frac{y}{\ell}\right), \end{aligned} \quad (\text{B7})$$

where

$$\begin{aligned} f_1(s) &= \int_{-\infty}^{\infty} \frac{dz}{2\pi} i(iz)^{-\alpha} \sin \frac{z}{2} e^{izs} \\ &= \frac{1}{2\Gamma(\alpha)} \left(\Theta\left(s + \frac{1}{2}\right) \left(s + \frac{1}{2}\right)^{\alpha-1} - \Theta\left(s - \frac{1}{2}\right) \left(s - \frac{1}{2}\right)^{\alpha-1} \right). \end{aligned} \quad (\text{B8})$$

Consequently, we obtain

$$b_z(x, y) = \frac{B}{\Gamma^2(\alpha)} \left(\Theta\left(x + \frac{\ell}{2}\right) \left(x + \frac{\ell}{2}\right)^{\alpha-1} - \Theta\left(x - \frac{\ell}{2}\right) \left(x - \frac{\ell}{2}\right)^{\alpha-1} \right) \left(\Theta\left(y + \frac{\ell}{2}\right) \left(y + \frac{\ell}{2}\right)^{\alpha-1} - \Theta\left(y - \frac{\ell}{2}\right) \left(y - \frac{\ell}{2}\right)^{\alpha-1} \right). \quad (\text{B9})$$

The phase difference is

$$\begin{aligned} \Delta\phi &= \frac{e}{\hbar} \int_{-a}^b dx \int_{-c}^d b_z(x, y) \\ &= \frac{eB}{\hbar\alpha^2\Gamma^2(\alpha)} b^{\alpha-1} d^{\alpha-1} \left(\left(1 + \frac{\ell}{2b}\right)^\alpha - \left(1 - \frac{\ell}{2b}\right)^\alpha \right) \left(\left(1 + \frac{\ell}{2d}\right)^\alpha - \left(1 - \frac{\ell}{2d}\right)^\alpha \right). \end{aligned} \quad (\text{B10})$$

In the limit $b \gg \ell$ and $d \gg \ell$,

$$\Delta\phi \approx \frac{eB\ell^2}{\hbar} \left(\frac{b^{\alpha-1} d^{\alpha-1}}{\Gamma^2(\alpha)} \right), \quad (\text{B11})$$

which is the limiting form given in the text.

2. Right Liouville Fractional Calculus

The resulting $b_z(\mathbf{k})$ is the same as Eq. (B7) but the function $f_1(s)$ is replaced with

$$\begin{aligned} f_2(s) &= \int_{-\infty}^{\infty} \frac{dz}{2\pi} i(-iz)^{-\alpha} \sin \frac{z}{2} e^{izs} \\ &= \frac{1}{2\Gamma(\alpha)} \left(\Theta(-s - \frac{1}{2})(-s - \frac{1}{2})^{\alpha-1} - \Theta(-s + \frac{1}{2})(-s + \frac{1}{2})^{\alpha-1} \right). \end{aligned} \quad (\text{B12})$$

Performing the area integral, we find that

$$\Delta\phi = \frac{eB}{\hbar\alpha^2\Gamma^2(\alpha)} a^{\alpha-1} c^{\alpha-1} \left(\left(1 + \frac{\ell}{2a}\right)^\alpha - \left(1 - \frac{\ell}{2a}\right)^\alpha \right) \left(\left(1 + \frac{\ell}{2c}\right)^\alpha - \left(1 - \frac{\ell}{2c}\right)^\alpha \right). \quad (\text{B13})$$

In the limit of $a \gg \ell$ and $c \gg \ell$,

$$\Delta\phi \approx \frac{eB\ell^2}{\hbar} \left(\frac{a^{\alpha-1} c^{\alpha-1}}{\Gamma^2(\alpha)} \right). \quad (\text{B14})$$

3. Feller Fractional Calculus

The resulting $b_z(\mathbf{k})$ is the same as Eq. (B7) but the function $f_1(s)$ is replaced with

$$\begin{aligned} f_3(s) &= \int_{-\infty}^{\infty} \frac{dz}{2\pi} \text{sgn}(z) |z|^{-\alpha} \sin \frac{z}{2} e^{izs} \\ &= -\frac{1}{4\Gamma(\alpha) \sin \frac{\pi\alpha}{2}} \left(\Theta(s + \frac{1}{2})(s + \frac{1}{2})^{\alpha-1} - \Theta(-s - \frac{1}{2})(-s - \frac{1}{2})^{\alpha-1} - \Theta(s - \frac{1}{2})(s - \frac{1}{2})^{\alpha-1} + \Theta(-s + \frac{1}{2})(-s + \frac{1}{2})^{\alpha-1} \right). \end{aligned} \quad (\text{B15})$$

The phase difference is

$$\begin{aligned} \Delta\phi &= \frac{eB}{4\hbar\alpha^2\Gamma^2(\alpha) \sin^2 \frac{\pi\alpha}{2}} \left(a^\alpha \left[\left(1 + \frac{\ell}{2a}\right)^\alpha - \left(1 - \frac{\ell}{2a}\right)^\alpha \right] + b^\alpha \left[\left(1 + \frac{\ell}{2b}\right)^\alpha - \left(1 - \frac{\ell}{2b}\right)^\alpha \right] \right) \\ &\quad \times \left(c^\alpha \left[\left(1 + \frac{\ell}{2c}\right)^\alpha - \left(1 - \frac{\ell}{2c}\right)^\alpha \right] + d^\alpha \left[\left(1 + \frac{\ell}{2d}\right)^\alpha - \left(1 - \frac{\ell}{2d}\right)^\alpha \right] \right), \end{aligned} \quad (\text{B16})$$

which in the limit of $a, b, c, d \gg \ell$, reduces to

$$\Delta\phi \approx \frac{eB\ell^2}{\hbar} \left(\frac{(a^{\alpha-1} + b^{\alpha-1})(c^{\alpha-1} + d^{\alpha-1})}{4\Gamma^2(\alpha) \sin^2 \frac{\pi\alpha}{2}} \right). \quad (\text{B17})$$

4. Riesz Fractional Calculus

The resulting $b_z(\mathbf{k})$ is the same as Eq. (B7) but the function $f_1(s)$ is replaced with

$$\begin{aligned} f_4(s) &= \int_{-\infty}^{\infty} \frac{dz}{2\pi} |z|^{-\alpha} \sin \frac{z}{2} e^{izs} \\ &= -\frac{i}{4\Gamma(\alpha) \cos \frac{\pi\alpha}{2}} \left(\Theta(s + \frac{1}{2})(s + \frac{1}{2})^{\alpha-1} + \Theta(-s - \frac{1}{2})(-s - \frac{1}{2})^{\alpha-1} - \Theta(s - \frac{1}{2})(s - \frac{1}{2})^{\alpha-1} - \Theta(-s + \frac{1}{2})(-s + \frac{1}{2})^{\alpha-1} \right). \end{aligned} \quad (\text{B18})$$

The phase difference is

$$\Delta\phi = \frac{eB}{4\hbar\alpha^2\Gamma^2(\alpha)\cos^2\frac{\pi\alpha}{2}} \left(a^\alpha \left[\left(1 + \frac{\ell}{2a}\right)^\alpha - \left(1 - \frac{\ell}{2a}\right)^\alpha \right] - b^\alpha \left[\left(1 + \frac{\ell}{2b}\right)^\alpha - \left(1 - \frac{\ell}{2b}\right)^\alpha \right] \right) \\ \times \left(c^\alpha \left[\left(1 + \frac{\ell}{2c}\right)^\alpha - \left(1 - \frac{\ell}{2c}\right)^\alpha \right] - d^\alpha \left[\left(1 + \frac{\ell}{2d}\right)^\alpha - \left(1 - \frac{\ell}{2d}\right)^\alpha \right] \right). \quad (\text{B19})$$

In the limit of $a, b, c, d \gg \ell$,

$$\Delta\phi \approx \frac{eB\ell^2}{\hbar} \left(\frac{(a^{\alpha-1} - b^{\alpha-1})(c^{\alpha-1} - d^{\alpha-1})}{4\Gamma^2(\alpha)\cos^2\frac{\pi\alpha}{2}} \right). \quad (\text{B20})$$

Appendix C: Fractional Aharonov-Bohm Effect of Disk Geometry

We consider now the disk geometry shown in Fig. (2). The magnetic field is given by

$$\mathbf{B} = B_z(\boldsymbol{\rho})\hat{z} = B\Theta(r - \rho)\hat{z}. \quad (\text{C1})$$

In Fourier space,

$$B_z(\mathbf{k}) = B \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-i\mathbf{k}\cdot\boldsymbol{\rho}} \Theta(r - \rho). \quad (\text{C2})$$

We now change to polar coordinates, $\boldsymbol{\rho} = \rho \cos \phi \hat{x} + \rho \sin \phi \hat{y}$ and $\mathbf{k} = k \cos \xi \hat{x} + k \sin \xi \hat{y}$. The result is

$$B_z(\mathbf{k}) = B \int_0^{2\pi} d\phi \int_0^r d\rho \rho e^{-ik\rho \cos(\phi - \xi)} \\ = \frac{2\pi r}{k} B J_1(kr) \quad (\text{C3})$$

1. Left Liouville Fractional Calculus

We perform the same calculation as in the rectangle case to obtain

$$b_z(\mathbf{k}) = B_z(\mathbf{k})(ik_x)^{1-\alpha}(ik_y)^{1-\alpha} \\ = 2\pi r B k^{1-2\alpha} J_1(kr)(i \cos \xi)^{1-\alpha}(i \sin \xi)^{1-\alpha}. \quad (\text{C4})$$

In position space,

$$b_z(\rho, \theta) = \frac{1}{4\pi^2} \int_0^\infty dk \int_0^{2\pi} d\xi e^{ik\rho \cos(\theta - \xi)} k^{2-2\alpha} J_1(kr)(i \cos \xi)^{\alpha-1}(i \sin \xi)^{\alpha-1}. \quad (\text{C5})$$

The phase difference is the area integral of b_z over the disk of radius R in Fig. (2),

$$\Delta\phi = \frac{e}{\hbar} \int_0^R d\rho \int_0^{2\pi} d\theta \rho b_z(\rho, \theta) \\ = \frac{erB}{2\pi\hbar} \int_0^\infty dk \int_0^{2\pi} d\xi \int_0^R d\rho \int_0^{2\pi} d\theta \rho k^{2-2\alpha} e^{ik\rho \cos(\theta - \xi)} J_1(kr)(i \cos \xi)^{\alpha-1}(i \sin \xi)^{\alpha-1}. \quad (\text{C6})$$

The θ integration yields

$$\int_0^{2\pi} d\theta e^{ik\rho \cos(\theta - \xi)} = 2\pi J_0(k\rho). \quad (\text{C7})$$

Consequently, we reduce the phase difference to

$$\Delta\phi = \frac{erB}{\hbar} \int_0^\infty dk \int_0^{2\pi} d\xi \int_0^R d\rho \rho k^{2-2\alpha} J_1(kr) J_0(k\rho) (i \cos \xi)^{\alpha-1} (i \sin \xi)^{\alpha-1}. \quad (\text{C8})$$

The integral over ρ can be done analytically,

$$\int_0^R d\rho \rho J_0(k\rho) = \frac{R}{k} J_1(kR). \quad (\text{C9})$$

Hence, the phase difference becomes

$$\Delta\phi = \frac{erRB}{\hbar} \int_0^\infty dk k^{1-2\alpha} J_1(kr) J_0(kR) \int_0^{2\pi} d\xi (i \cos \xi)^{\alpha-1} (i \sin \xi)^{\alpha-1}. \quad (\text{C10})$$

The two integrals can be evaluated as

$$\int_0^\infty dk k^{1-2\alpha} J_1(kr) J_0(kR) = \frac{2^{1-2\alpha} r R^{2\alpha-3} \Gamma(2-\alpha)}{\Gamma(\alpha)} {}_2F_1(1-\alpha, 2-\alpha; 2; (\frac{r}{R})^2) \quad (\text{C11})$$

and

$$\int_0^{2\pi} d\xi (i \cos \xi)^{\alpha-1} (i \sin \xi)^{\alpha-1} = \frac{2^\alpha \sin^2 \frac{\pi\alpha}{2} \sqrt{\pi} \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2})}. \quad (\text{C12})$$

Here ${}_2F_1(a, b; c; z)$ is a hypergeometric function. Finally, the phase difference is

$$\Delta\phi = \frac{e}{\hbar} \pi r^2 B R^{2\alpha-2} \left(\frac{\sqrt{\pi} 2^{1-\alpha} \Gamma(2-\alpha) \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\alpha) \Gamma(\frac{3}{2} - \frac{\alpha}{2})} \sin^2 \frac{\pi\alpha}{2} {}_2F_1(1-\alpha, 2-\alpha; 2; \frac{r^2}{R^2}) \right) \quad (\text{C13})$$

The terms in the parenthesis reduce to 1 in the limit $\alpha \rightarrow 1$.

2. Right Liouville Fractional Calculus

The phase difference from this fractional calculus is the same as the phase in Eq. 31 because one can show that

$$b_z(\mathbf{k}) = 2\pi r B k^{1-2\alpha} J_1(kr) (-i \cos \xi)^{1-\alpha} (-i \sin \xi)^{1-\alpha} \quad (\text{C14})$$

and the integral

$$\int_0^{2\pi} d\xi (-i \cos \xi)^{\alpha-1} (-i \sin \xi)^{\alpha-1} = \frac{2^\alpha \sin^2 \frac{\pi\alpha}{2} \sqrt{\pi} \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2})}. \quad (\text{C15})$$

3. Feller Fractional Calculus

For this definition, one can show that

$$b_z(\mathbf{k}) = 2\pi r B k^{1-2\alpha} J_1(kr) |\cos \xi|^{1-\alpha} |\sin \xi|^{1-\alpha}. \quad (\text{C16})$$

The only difference from the right Liouville calculus is the integration over ξ . One finds

$$\int_0^{2\pi} d\xi |\cos \xi|^{1-\alpha} |\sin \xi|^{1-\alpha} = \frac{2^\alpha \sqrt{\pi} \Gamma(1 - \frac{\alpha}{2})}{\Gamma(\frac{3}{2} - \frac{\alpha}{2})}. \quad (\text{C17})$$

And hence the phase difference is

$$\Delta\phi = \frac{e}{\hbar}\pi r^2 B R^{2\alpha-2} \left(\frac{\sqrt{\pi} 2^{1-\alpha} \Gamma(2-\alpha) \Gamma(1-\frac{\alpha}{2})}{\Gamma(\alpha) \Gamma(\frac{3}{2}-\frac{\alpha}{2})} {}_2F_1(1-\alpha, 2-\alpha; 2; \frac{r^2}{R^2}) \right). \quad (\text{C18})$$

4. Riesz Fractional Calculus

For this definition, one can show that

$$b_z(\mathbf{k}) = -2\pi r B k^{1-2\alpha} J_1(kr) \cos \xi |\cos \xi|^{-\alpha} \sin \xi |\sin \xi|^{-\alpha}. \quad (\text{C19})$$

The integral over ξ vanishes because $\cos \xi |\cos \xi|^{-\alpha} \sin \xi |\sin \xi|^{-\alpha}$ is an odd function. As a result

$$\Delta\phi = 0. \quad (\text{C20})$$

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- [1] P. Phillips and C. Chamon, Phys. Rev. Lett. **95**, 107002 (2005).
 - [2] S. A. Hartnoll and A. Karch, Phys. Rev. B **91**, 155126 (2015), arXiv:1501.03165 [cond-mat.str-el].
 - [3] D. v. d. Marel, H. J. A. Molegraaf, J. Zaanen, Z. Nussinov, F. Carbone, A. Damascelli, H. Eisaki, M. Greven, P. H. Kes, and M. Li, Nature **425**, 271 (2003).
 - [4] D. N. Basov, R. D. Averitt, D. van der Marel, M. Dressel, and K. Haule, Rev. Mod. Phys. **83**, 471 (2011).
 - [5] Y. Zhang, N. P. Ong, Z. A. Xu, K. Krishana, R. Gagnon, and L. Taillefer, Phys. Rev. Lett. **84**, 2219 (2000).
 - [6] T. R. Chien, Z. Z. Wang, and N. P. Ong, Phys. Rev. Lett. **67**, 2088 (1991).
 - [7] K. Limtragoon and P. Phillips, Phys. Rev. B **92**, 155128 (2015), arXiv:1506.00649 [cond-mat.str-el].
 - [8] A. Karch, K. Limtragoon, and P. W. Phillips, ArXiv e-prints (2015), arXiv:1511.02868 [cond-mat.str-el].
 - [9] D. J. Gross, *Methods in Field Theory: Les Houches 1975*, edited by R. Balian and J. Zin-Justin, p. 181 (North-Holland, 1975).
 - [10] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley (now Perseus Books), 1995).
 - [11] X.-G. Wen, Physical Review B **46**, 2655 (1992).
 - [12] B. Gout  raux and E. Kiritsis, Journal of High Energy Physics **4**, 53 (2013), arXiv:1212.2625 [hep-th].
 - [13] B. Gout  raux, Journal of High Energy Physics **1**, 80 (2014), arXiv:1308.2084 [hep-th].
 - [14] A. Karch, Journal of High Energy Physics **7**, 21 (2015), arXiv:1504.02478 [hep-th].
 - [15] Y. Ando, S. Ono, X. F. Sun, J. Takeya, F. F. Balakirev, J. B. Betts, and G. S. Boebinger, Phys. Rev. Lett. **92**, 247004 (2004).
 - [16] F. F. Balakirev, J. B. Betts, A. Migliori, S. Ono, Y. Ando, and G. S. Boebinger, Nature **424**, 912 (2003).
 - [17] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
 - [18] M. J. Lazo, Physics Letters A **375**, 3541 (2011), arXiv:1108.3493 [math-ph].
 - [19] R. Herrmann, Physics Letters A **372**, 5515 (2008), arXiv:0708.2262 [math-ph].
 - [20] S. K. Domokos and G. Gabadadze, ArXiv e-prints (2015), arXiv:1509.03285 [hep-th].
 - [21] E. C. Grigoletto and E. C. de Oliveira, Applied Mathematics **4** (2013).
 - [22] K. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional differential Equations* (John Wiley & Sons, Inc., New York, 1993).
 - [23] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, Amsterdam, 2006).
 - [24] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications* (Gordon and Breach Science Publishers, Amsterdam, 1993).
 - [25] O. P. Agrawal, Journal of Mathematical Analysis and Applications **272**, 368 (2002).